Theory of creeping gravity currents of a non-Newtonian liquid

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Recently several experiments on creeping gravity currents have been performed, using highly viscous silicone oils and putties. The interpretation of the experiments relies on the available theoretical results that were obtained by means of the lubrication approximation with the assumption of a Newtonian rheology. Since very viscous fluids are usually non-Newtonian, an extension of the theory to include non-Newtonian effects is needed. We derive the governing equations for unidirectional and axisymmetric creeping gravity currents of a non-Newtonian liquid with a power-law rheology, generalizing the usual lubrication approximation. The equations differ from those for Newtonian liquids, being nonlinear in the spatial derivative of the thickness of the current. Similarity solutions for currents whose volume varies as a power of time are obtained. For the spread of a constant volume of liquid, analytic solutions are found that are in good agreement with experiment. We also derive solutions of the waiting-time type, as well as those describing steady flows from a constant source to a sink. General traveling-wave solutions are given, and analytic formulas for a simple case are derived. A phase plane formalism that allows the systematic derivation of self-similar solutions is introduced. The application of the Boltzmann transform is briefly discussed. All the self-similar solutions obtained here have their counterparts in Newtonian flows, as should be expected because the power-law rheology involves a single-dimensional parameter as the Newtonian constitutive relation. Thus one finds similarity solutions whenever the analogous Newtonian problem is self-similar, but now the spreading relations are rheology-dependent. In most cases this dependence is weak but leads to significant differences easily detected in experiments. The present results may also be of interest for geophysics since the lithosphere deforms according to an average power-law rheology. [S1063-651X(99)09011-X]

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I. INTRODUCTION

Gravity currents are ubiquitous phenomena that occur in many situations of scientific and engineering interest [1,2]. Various regimes are possible, according to the relative magnitude of the forces acting on a typical fluid element. Of considerable practical interest is the viscosity dominated regime called creeping flow. In this regime the motion is nearly horizontal and very slow, so that inertia effects are negligible and the flow is governed by a balance between gravity and the viscous forces. The flows of thin layers of highly viscous fluids on a horizontal surface, as well as certain magma flows, belong to this class. Recently several experiments on creeping gravity currents have been performed (see, for example, [3–6]). The theory of these flows [7–10] has been based on the lubrication theory approximation [11] with the assumption that the fluid is Newtonian. It is not trivial what changes will result in the theory if the fluid is non-Newtonian, as is the case of many highly viscous liquids of practical interest. We notice that the silicone oils used in the above-mentioned experiments as well as the silicone putties used in the analogic modeling of gravity flows of the earths crust (see [12]) have a non-Newtonian behavior [13,14,4]. Also, a non-Newtonian constitutive relation is required to describe the rheology of the lithosphere (see, for example, [15]). We notice that the deformations of the lithosphere associated with orogeny can also be described in terms of creeping gravity currents. Scaling laws that describe the time evolution of mountain belts were derived [16] considering the combined effect of crustal shortening, isostasy, and creeping gravity flow at the root of the belt. Then the extension of the theory that we presently develop is of considerable practical interest.

One of the simplest non-Newtonian models is based on the so-called power-law constitutive relation of the form [17,18]

$$\tau_{ij} = A E^{(1-\lambda)/\lambda} \dot{\varepsilon}_{ij}, \tag{1}$$

in which τ_{ij} is the deviatoric stress tensor, $\dot{\varepsilon}_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2$ is the strain rate, and

$$E = (\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij})^{1/2} \tag{2}$$

is the second invariant of the strain rate tensor (A and the rheological index λ are constants). The formula (1) is sometimes known as Ostwald's or Norton's constitutive relation. A power-law rheology such as Eq. (1) is usually accepted as a good description of the vertically averaged mechanical properties of the lithospheric rocks (see, for example, [19]), and is adequate to describe the behavior of many non-Newtonian liquids within appropriate ranges of the strain rate, like those of the above-mentioned experiments.

In this paper we develop a theory of the creeping gravity currents of a liquid that obeys the power-law constitutive relation (1). We use an approximation (analogous to the lubrication approximation for Newtonian liquids) to derive equations for a current flowing on a rigid horizontal surface, and show that the thickness of the current satisfies a nonlinear parabolic differential equation that is a generalization of the nonlinear diffusion equation usually called the porous media equation in the mathematical literature. Next we derive some similarity solutions that describe the flows corresponding to various initial and boundary conditions; these include (i) the spread of a constant volume of the liquid, (ii) the currents produced by sources located at the origin, (iii) the steady flow from a source to a sink, and (iv) solutions of the waiting-time type (analogous to those arising in nonlinear diffusion; see, for example, [20,21], also [22,23] for theoretical details, and [4,5] for experiments). In Cartesian geometry, the system also allows propagating wave solutions. Finally we set up a phase plane formalism that allows us to investigate systematically the entire family of self-similar solutions of the governing equations. We also discuss the Boltzmann transform method of solution. A detailed investigation of the solutions corresponding to the integral curves in the phase plane is left for future work.

II. THE "LUBRICATION APPROXIMATION" FOR A NON-NEWTONIAN LIQUID

The governing equations of the creeping gravity flow of a power-law liquid on a rigid horizontal surface are obtained starting from the following assumptions: (i) the motion is essentially horizontal (so that the vertical component of the velocity is negligibly small), (ii) inertia effects are negligible, and (iii) the length of the current is much larger than its depth. These assumptions imply a purely hydrostatic pressure. In this paper we shall consider only planar and axisymmetric flows, i.e., flows that depend on a single horizontal coordinate (Cartesian for planar symmetry, radial for axial symmetry). The horizontal coordinate is x, the vertical coordinate is z, and t denotes the time. The acceleration of gravity is g and the constitutive relation (1) is assumed.

With these assumptions, it can be easily shown that the x and z components of the momentum equation can be approximated as

$$\frac{\partial p}{\partial x}-2^{(\lambda-1)/2\lambda}\sigma A\frac{\partial}{\partial z}\left(\frac{\partial|v_x|}{\partial z}\right)^{1/\lambda}=0,\quad \sigma=\mathrm{sgn}(v_x) \tag{3}$$

and

$$\frac{\partial p}{\partial z}+\rho g=0. \tag{4}$$

In Eqs. (3) and (4), ρ is the density, p the pressure, and $v_x(x,z,t)$ is the horizontal component of the velocity. Notice that strictly speaking $\sigma=\mathrm{sgn}(\partial v_x/\partial z)$, but in our system it coincides with $\sigma=\mathrm{sgn}(v_x)$. In deriving Eqs. (3) and (4) we have assumed that the strongest variation of v_x is in z, and have neglected the x variation of v_x (which is crucial for the continuity equation). In fact we shall sneak in the x dependence through the boundary condition at the free surface $z=h(x,t)$, where $h(x,t)$ denotes the thickness of the current.

Integrating Eq. (4) with the condition that at $z=h$, $p=0$, we obtain

$$p=\rho g(h-z), \tag{5}$$

which gives the required slow x dependence to the left-hand side of Eq. (3), i.e.,

$$\frac{\partial p}{\partial x}=\rho g\frac{\partial h}{\partial x}. \tag{6}$$

Within the context of the preceding discussion, and with the boundary conditions (i) no slip at the bottom $z=0$ and (ii) no tangential stress at $z=h$, we can readily integrate Eq. (3) to find

$$v_x=\frac{\lambda+2}{\lambda+1}v\left[1-\left(1-\frac{z}{h}\right)^{\lambda+1}\right], \tag{7}$$

where

$$v=\langle v_x\rangle=\frac{1}{h}\int_0^h v_x dz=2^{(1-\lambda)/2\lambda}\sigma\left(\frac{\rho g}{A}\right)^{\lambda}\frac{h^{\lambda+1}}{\lambda+2}\left(-\sigma\frac{\partial h}{\partial x}\right)^{\lambda} \tag{8}$$

is the vertically averaged speed. From Eq. (8) it is obvious that the sign of v, as expected, is always opposite to that of $\partial h/\partial x$. The explicit appearance of σ is a necessary consequence of the non-Newtonian rheology.

Equation (8) represents the essence of the momentum transfer equation for this paper. We now take the vertically averaged continuity equation

$$\frac{\partial h}{\partial t}+\boldsymbol{\nabla}\cdot(\mathbf{e}_x\langle v_x h\rangle)=\frac{\partial h}{\partial t}+\boldsymbol{\nabla}\cdot(\mathbf{e}_x v h)=0 \tag{9}$$

and combine it with Eq. (8) to give a single equation:

$$\frac{\partial w}{\partial t}+\sigma x^{-n}\frac{\partial}{\partial x}\left[x^{-n}w^{\lambda+2}\left(-\sigma\frac{\partial w}{\partial x}\right)^{\lambda}\right]=0, \tag{10}$$

where we have defined the "reduced thickness" w:

$$w=a_{\lambda}^{-1}h, \tag{11}$$

with

$$a_{\lambda}=\left(2^{(1-\lambda)/2\lambda}(\lambda+2)^{1/\lambda}\frac{A}{\rho g}\right)^{\lambda/(2\lambda+1)}, \tag{12}$$

in order to absorb the parameters A, ρ, and g. Notice also that the index $n=0$ ($n=1$) signifies Cartesian (axial) symmetry.

Equation (10), or equivalently the set of equations (8) and (9), rewritten in terms of w, i.e.,

$$v-\sigma w^{\lambda+1}\left(-\sigma\frac{\partial w}{\partial x}\right)^{\lambda}=0, \tag{13a}$$

$$\frac{\partial w}{\partial t}+x^{-n}\frac{\partial}{\partial x}(x^n w v)=0, \tag{13b}$$

are the governing equations for creeping gravity flows in this generalized lubrication approximation, as they reduce to the usual formulas for a Newtonian fluid.

Equation (10) is a nonlinear parabolic equation of diffusive type that is different from the usual equations of nonlinear diffusion (see, for example, [24]). It can be observed that the assumption of non-Newtonian rheology ($\lambda \neq 1$) introduces a nonlinearity in the spatial derivative of w that was not present in the Newtonian case, so that our generalization is not at all trivial. However, we shall show in the following sections that many solutions of Eq. (10) are closely analogous to solutions pertaining to Newtonian liquids. It will be also shown that in many instances the currents are characterized by a sharp, well defined front, the current extends up to a certain value $x = x_f$, and as $x \to x_f$ the thickness h vanishes, there being no fluid ahead of the front. In this connection, it will be seen that the present lubrication approximation predicts profiles of the form $h \propto X^s$, with $X = |x - x_f|$ and $0 < s < 1$. Naturally these vertical profiles are incorrect near the front, where the approximation breaks down. The same problem arises in the context of Newtonian liquids and in this case it has been shown (see the lengthy discussions of [7,8]) that the model describes correctly the general shape and dynamics of the currents, regardless of the fact that the vertical fronts are surely not realistic. We see no reason why the situation should be different in the present case. Actually various experiments with Newtonian [25,26,3,9] and non-Newtonian liquids [4,14,6] show that the lubrication approximation describes surprisingly well the motion of the front and the profiles of the currents, even quite close to the front. Accordingly, we shall accept solutions with sharp fronts, subject to the qualification that there exists a certain small region near the front where their profiles differ markedly from the true solutions.

III. SIMILARITY SOLUTIONS

By making a judicious choice for w, we are left with no parameters in the governing equations. The variables w and v, in fact, have dimensions that can be completely specified in terms of length $[L]$ and time $[T]$. We can take advantage of this fact and express w and v in terms of two dimensionless phase variables Z and V,

$$w = (x^{\lambda+1} t^{-1} Z)^{1/(2\lambda+1)}, \quad v = x t^{-1} V, \tag{14}$$

which, in general, depend on x, t, and the parameters of the problem that enter into its specific initial and boundary conditions. Substituting Eq. (14) into Eq. (13), one finds

$$\sigma\left(\sigma \frac{V}{Z}\right)^{1/\lambda} + \frac{1}{\lambda_2}\left(\lambda_1 + \frac{x}{Z}\frac{\partial Z}{\partial x}\right) = 0 \tag{15}$$

and

$$\frac{t}{Z}\frac{\partial Z}{\partial t} = 1 - [\lambda_2(n+1) + \lambda_1]V - V\frac{x}{Z}\frac{\partial Z}{\partial x} - \lambda_2 x \frac{\partial V}{\partial x}, \tag{16}$$

with the definitions

$$\lambda_1 = \lambda + 1, \quad \lambda_2 = 2\lambda + 1. \tag{17}$$

Let us now assume that the problem involves only one parameter, b, with independent dimensions. Clearly, it can be assumed without loss of generality that

$$[b] = L T^{-\delta}, \tag{18}$$

where δ is a numerical constant. Then there will be a single dimensionless combination of x, t, and b, which we can take as

$$\zeta = x / b t^{\delta}. \tag{19}$$

In this case $Z = Z(\zeta)$, $V = V(\zeta)$, and the motion is self-similar, ζ being the similarity variable. For self-similar flows the phase variables Z and V satisfy the following ordinary differential equations:

$$\sigma\left(\sigma \frac{V}{Z}\right)^{1/\lambda} + \frac{1}{\lambda_2}\left(\lambda_1 + \frac{\zeta}{Z}\frac{dZ}{d\zeta}\right) = 0, \tag{20}$$

$$\lambda_2 \zeta \frac{dV}{d\zeta} = 1 - [\lambda_2(n+1) + \lambda_1]V - (V - \delta)\frac{\zeta}{Z}\frac{dZ}{d\zeta}. \tag{21}$$

Later on we shall indicate how to obtain from Eqs. (20) and (21) a general formalism that allows us to derive (in a systematic way) the entire family of solutions corresponding to the similarity variable ζ. In the rest of the present section, we shall discuss some special solutions of particular interest.

A. Creeping gravity currents whose volume varies with time according to a power law

These flows obey the global continuity equation

$$\int_0^{x_f(t)} (2\pi x)^n h(x,t)\, dx = q_\alpha t^\alpha, \tag{22}$$

where $q_\alpha = \text{const}$ and x_f denotes the position of the front. Clearly $\alpha = 0$ corresponds to a volume conserving current, $\alpha = 1$ to a source of constant flux at $x = 0$, etc. For Newtonian liquids, these flows have been studied already [7].

Using Eqs. (9) and (14) in Eq. (22) one finds

$$\delta = \frac{1 + \lambda_2 \alpha}{\lambda_2(n+1) + \lambda_1} \tag{23}$$

and

$$b = \left(\frac{q_\alpha}{a_\lambda}\right)^{\beta}, \quad \beta = \frac{\lambda_2}{\lambda_2(n+1) + \lambda_1}, \tag{24}$$

$$\zeta_f = \left[(2\pi)^n \int_0^1 \eta^{1/\beta - 1} Z^{1/\lambda_2} d\eta\right]^{-\beta}, \tag{25}$$

with

$$\eta = \zeta / \zeta_f. \tag{26}$$

From these results we can determine the spreading relations for these currents: the equation of motion of the front is given by

$$x_f(t) = \zeta_f b t^{\delta}, \tag{27}$$

and for fixed $\eta = x/x_f$ the thickness of the current varies as

$$h \propto t^{\gamma}, \quad \gamma = \delta[\alpha\lambda_1 - (n+1)] \tag{28}$$

and the average flow velocity as

$$v \propto t^{\delta-1}. \tag{29}$$

To determine the profile of the current and the dependence of the average flow velocity on η, it is of course necessary to solve Eqs. (20) and (21). Barring a few special cases, it is not possible to obtain close form solutions. For a very important case, i.e., that of a volume conserving current ($\alpha = 0$, implying the spreading of a constant volume of liquid), Eqs. (20) and (21) admit a special close form solution given by ($\sigma = 1$)

$$Z = \delta\left[\frac{\lambda_2}{\lambda_1}(\eta^{-\lambda_1/\lambda} - 1)\right]^{\lambda}, \quad V = \delta, \tag{30}$$

with $\delta = [\lambda_2(n+1) + \lambda_1]^{-1}$, $\beta = \lambda_2\delta$, from which one derives

$$h = [a_\lambda^{n+1} q_0^{\lambda_1/\lambda_2}]^{\beta}\left[\delta\zeta_f^{\lambda_1}\left(\frac{\lambda_2}{\lambda_1}\right)^{\lambda}\right]^{1/\lambda_2} t^{-\delta(n+1)}(1 - \eta^{\lambda_1/\lambda})^{\lambda/\lambda_2}, \tag{31}$$

$$v = \delta\frac{x}{t}, \tag{32}$$

with

$$\zeta_f = \left\{ (2\pi)^n\left[\delta\left(\frac{\lambda_2}{\lambda_1}\right)^{\lambda}\right]^{1/\lambda_2} \frac{\Gamma\left(1 + \frac{\lambda}{\lambda_2}\right)\Gamma\left(\frac{1+\lambda(2+n)}{\lambda_1}\right)}{(1+n)\Gamma\left(1 + \frac{\lambda}{\lambda_2} + \frac{\lambda(1+n)}{\lambda_1}\right)} \right\}^{-\beta}. \tag{33}$$

It can be verified that for $\lambda = 1$ these solutions reduce to those previously known (see, for instance, [7]). The self-similarity exponent δ depends weakly on the rheology. In Fig. 1 we show the profiles of constant-volume currents ($\alpha = 0$) for different values of λ in the case $n = 0$.

B. Waiting-time solutions

The governing equations admit what are called "waiting-time solutions." These solutions represent flows whose front does not move during a finite interval of time, although there is liquid movement behind the front. Solutions of this type appear in problems of nonlinear diffusion, nonlinear heat conduction, and other related problems (see, for example, [20,21]). Waiting-time Newtonian creeping gravity currents have been studied theoretically [8,10] and experimentally [13,4,5]. They are related to a singular solutions of Eqs. (20) and (21) of the form $Z = \text{const}$, $V = \text{const}$. It can be easily verified that, in the present case of non-Newtonian rheology, one also finds solutions of this kind.

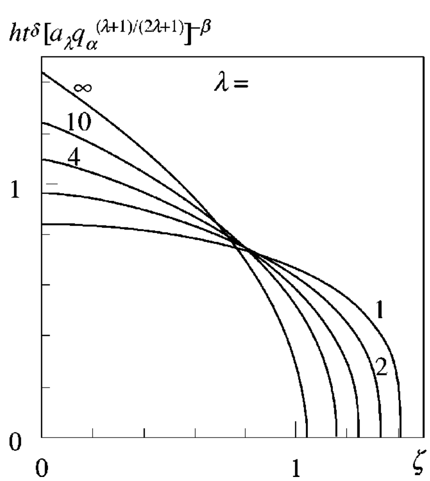

FIG. 1. Profiles of constant-volume currents for different λ.

There is, in effect, a single constant solution of the system (20) and (21), given by

$$V = V_0 = \frac{1}{\lambda_2(n+1) + \lambda_1}, \tag{34a}$$

$$Z = Z_0 = -V_0\left(\frac{\lambda_2}{\lambda_1}\right)^{\lambda}, \tag{34b}$$

for $\sigma = -1$. The corresponding flow is given by

$$h = a_\lambda\left(\frac{x^{\lambda_1}Z_0}{t}\right)^{1/\lambda_2}, \quad v = V_0\frac{x}{t}, \quad t < 0. \tag{35}$$

This solution is only valid for negative t, and blows up at $t = 0$. It represents a current whose front is stationary for a finite time. In the analogous case of the usual nonlinear diffusion equation, it has been shown [22,23,10] how to construct solutions of this type that can be extended to positive time (when the front starts to move). Equation (34) is not the only waiting-time current that can occur for non-Newtonian liquids; the discussion of initial conditions leading to the waiting-time behavior will be given below.

IV. STEADY FLOWS

It is easy to verify that the governing equations (13) admit a time-independent solution. It yields a current given by

$$h = h_0(1 - \eta^{(\lambda-n)/\lambda})^{\lambda/(2\lambda+2)}, \tag{36}$$

$$v = (h_0/a_\lambda)^{\lambda_2} x_0^{-1}\eta^{-n}(1 - \eta^{(\lambda-n)/\lambda})^{-\lambda/(2\lambda+2)}, \tag{37}$$

$$n \neq 1, \quad \eta = x/x_0, \quad x_0 = \text{const}, \quad h_0 = \text{const}, \tag{38}$$

which represents the flow over a horizontal surface of finite extent, having an edge at $x = x_0$. The liquid flows from a constant source at $x = 0$ to the edge, and spills over it. It can be verified that

$$q = (2\pi x)^n v h = (2\pi)^n x_0^{n-\lambda} h_0^{2\lambda+2} a_\lambda^{-\lambda_2} = \text{const}. \tag{39}$$

These solutions are analogous to those corresponding to the Newtonian case [8]; in this reference the connection between the steady-state solutions and the self-similar ones is discussed. In the case $n = 1$, $\lambda = 1$ (Newtonian), Eqs. (36) and (37) are not valid and w, v depend logarithmically on x.

V. TRAVELING-WAVE SOLUTIONS

For the special case of Cartesian geometry ($n = 0$) one can find traveling-wave solutions of Eq. (13). Let

$$w = w(\xi), \quad \xi = ct - x, \quad c = \text{const}. \tag{40}$$

In the present case, of course, c does not depend on the properties of the fluid, but is a parameter determined by the boundary conditions, for instance a piston moving at a constant speed that pushes the liquid. As a consequence, it may assume any value. Using Eq. (40), Eq. (21) can be integrated obtaining

$$\xi - \xi_0 = \sigma \int \left[\frac{\sigma w^{\lambda+2}}{c(w-K)} \right]^{1/\lambda} dw, \tag{41}$$

where K and ξ_0 are arbitrary constants. Thus the problem has been reduced to a quadrature. As a special case, consider $K = 0$. Let us further assume $c > 0$; then $\sigma = 1$. Evaluating Eq. (41), one obtains

$$w = \left[\frac{\lambda_2}{\lambda} c^{1/\lambda} (\xi - \xi_0) \right]^{\lambda/\lambda_2}. \tag{42}$$

This solution is the generalization for non-Newtonian liquids of the traveling-wave solution (see [8]) already known for Newtonian flows. It represents a current that advances with constant speed c on an infinite horizontal supporting plate; its front is located at $x = ct - \xi_0$.

If one assumes $c < 0$ (which requires $\sigma = -1$), a wave traveling in the opposite sense is obtained; we omit details for brevity.

These currents describe the flow produced by a plane piston, or by a spatula, that advances steadily, pushing a constant volume of liquid in front of it. Actually, the present approximation ceases to be valid immediately in front of the piston, so that our formula is a good description of the profile of the current as long as one considers only the flow far from the piston.

VI. OUTLINE OF A PHASE PLANE FORMALISM

The entire family of self-similar solutions of the type (13) and (17) can be systematically investigated by means of a phase plane formalism developed in analogy to that of gas dynamics (see [27,28]). In this paper, we closely follow the treatment given in [8]. Starting from the coupled equations (20) and (21), it is possible to eliminate ζ to obtain an autonomous first-order ordinary differential equation for $V(Z)$:

$$\frac{dV}{dZ} = \frac{N(V,Z)}{D(V,Z)}, \tag{43}$$

where

$$N(V,Z) = (\delta - V)[\lambda_2 \sigma (\sigma V/Z)^{1/\lambda} + \lambda_1] + [\lambda_2(n-1) + \lambda_1]V - 1 \tag{44}$$

and

$$D(V,Z) = \lambda_2 Z [\lambda_2 \sigma (\sigma V/Z)^{1/\lambda} + \lambda_1]. \tag{45}$$

Once Eq. (43) has been solved, and $V(Z)$ is known, $\zeta(Z)$ can be obtained from

$$\frac{d(\ln \zeta)}{dZ} = -\frac{\lambda_2}{D(V,Z)}, \tag{46}$$

by means of a simple quadrature.

Thus the essential step in the solution of a self-similar problem is the integration (numerical, in general) of Eq. (43). The Z-V plane is usually called the ''phase plane''; a solution of Eq. (43) is represented by a curve in the phase plane, which is called an integral curve. A single integral curve passes through any regular point of the phase plane. Any integral curve represents a self-similar current of a certain sort. A self-similar solution characterized by specific boundary conditions is represented by one or more pieces of the appropriate integral curves. As an example, the integral curve corresponding to the self-similar current that describes the spread of a constant volume of liquid is given by the straight line $V = \delta$ (see Sec. III A).

To determine which integral curve corresponds to the given problem, it is necessary to investigate the behavior of $V(Z)$ in the neighborhood of the singular points of Eq. (43). The whole (Z,V) plane needs to be considered, and according to Eq. (14), solutions with $Z > 0$ correspond to $t > 0$, while solutions in the half-plane $Z < 0$ are meaningful only for $t < 0$. It can also be observed that $\sigma = 1$ in the first ($Z > 0, V > 0$) and third ($Z < 0, V < 0$) quadrants, and $\sigma = -1$ in the second ($Z < 0, V > 0$) and fourth ($Z > 0, V < 0$). The singular points themselves also represent solutions: the waiting-time solution, described in Sec. III B, is an example. The detailed investigation of the entire family of self-similar flows is left for future work.

VII. ADVANCING FRONTS AND WAITING-TIME BEHAVIOR

An important singular point of Eq. (43) is $\mathcal{A}$ ($V_{\mathcal{A}} = \delta, Z_{\mathcal{A}} = 0$). Two integral curves pass through $\mathcal{A}$, which is a saddle. One $Z = 0$ is uninteresting; the other is given approximately by

$$V \cong \delta + \sigma\lambda \frac{[\lambda_2(n+1) + \lambda_1]\delta - 1}{\lambda_2(3\lambda+1)} \left(\frac{\sigma Z}{\delta} \right)^{1/\lambda} \tag{47}$$

in the neighborhood of $\mathcal{A}$. The point $\mathcal{A}$ represents the advancing front of a current. Using Eq. (47), we integrate Eq. (46) and find

$$Z \cong \delta\left[\frac{\lambda_2}{\lambda}(1-\eta)\right]^{\lambda} \tag{48}$$

and

$$h=\left[2^{\lambda_1/2}(\lambda+2)\,\delta\left(\frac{\lambda_2}{\lambda}\right)^{\lambda}\frac{A}{\rho g}\right]^{1/\lambda_2}\left(\frac{x_f}{t}X^{\lambda}\right)^{1/\lambda_2}, \tag{49}$$

$$v=\delta\frac{x_f}{t}\eta, \tag{50}$$

with $X=|x-x_f|$. One thus finds that an advancing front has a characteristic profile of the form

$$h \propto X^{\lambda/\lambda_2}. \tag{51}$$

It is interesting to find that the shape of an advancing front depends only on the rheological index λ, and not on any other parameter of the problem.

It can be easily shown from Eq. (13) that *any* front that advances with a finite, nonvanishing speed (and not only those of the self-similar currents) must have the unique shape (51). This fact is related to the existence of waiting-time solutions (see [23], also [13] for a discussion of this problem in the context of Newtonian liquids): it can be shown that if the initial profile of the current near a front is of the form

$$h(t=0) \propto X^{s}, \quad s=\text{const}, \tag{52}$$

several possibilities arise (of course, in general, the motion will not be self-similar at the beginning): (i) if $0<s<\lambda_1/\lambda_2$, the front begins to move at once with (a) infinite velocity if $0<s<\lambda/\lambda_2$, (b) finite velocity if $s=\lambda/\lambda_2$, or zero velocity if $\lambda/\lambda_2<s<\lambda_1/\lambda_2$; (ii) if $s=\lambda_1/\lambda_2$, the front begins to move after a finite waiting time t_w, (iii) if $s>\lambda_1/\lambda_2$, the front also begins to move after a finite waiting time t_w, but before that a discontinuity of the slope of the profile develops behind the front (a corner shock). This corner shock (more precisely, corner layer) advances towards the front, which does not move until it is overtaken by the corner shock at $t=t_w$. The experiments and the numerical solutions of the governing equations support these conclusions for the Newtonian case [5,10].

VIII. BOLTZMANN TRANSFORM FORMALISM

An alternative way of deriving solutions of the governing equation (10) is based on seeking solutions of the form

$$w=t^{\mu}f(\vartheta), \quad \vartheta=x/t^{\delta}. \tag{53}$$

Substituting Eq. (53) into Eq. (10), one finds that the condition

$$\delta\lambda_1=\mu\lambda_2+1 \tag{54}$$

must be satisfied for Eq. (53) to be a solution. The resulting ordinary differential equation takes the form

$$\vartheta^{n}(\mu f-\delta\vartheta f')+\sigma[\vartheta^{n}f^{\lambda+2}(-\sigma f')^{\lambda}]'=0, \tag{55}$$

where the prime denotes the derivative with respect to ϑ. If we choose

$$\mu=-\delta(n+1), \tag{56}$$

Eq. (54) can be integrated once to give

$$\sigma\vartheta^{n}f^{\lambda+2}(-\sigma f')^{\lambda}=\delta\vartheta^{n+1}f+\text{const}, \tag{57}$$

along with an expression for δ in terms of the parameters of the system,

$$\delta=[\lambda_2(n+1)+\lambda_1]^{-1}. \tag{58}$$

For $\text{const}=0$, Eq. (57) can be readily integrated to yield

$$f=\left[\Lambda-\sigma(\sigma\delta)^{1/\lambda}\frac{\lambda_2}{\lambda_1}\vartheta^{(\lambda+1)/\lambda}\right]^{\lambda/\lambda_2}, \tag{59}$$

where Λ is a constant of integration.

Remembering the fact that our original system is invariant under time translation (and also under space translation for the $n=0$ case), we realize that

$$w=\tau^{-\delta(n+1)}\left[\Lambda-\sigma(\sigma\delta)^{1/\lambda}\frac{\lambda_2}{\lambda_1}\left(\frac{x}{\tau^{\delta}}\right)^{(\lambda+1)/\lambda}\right]^{\lambda/\lambda_2}, \tag{60}$$

where $\tau=t+\Delta$ (and Δ denotes some arbitrary time) is also a solution. Equation (60) can be viewed as giving the time evolution of the system starting from a well-defined initial condition at $t=0$,

$$w_0=\Delta^{-\delta(n+1)}\left[\Lambda-\sigma(\sigma\delta)^{1/\lambda}\frac{\lambda_2}{\lambda_1}\left(\frac{x}{\Delta^{\delta}}\right)^{\lambda_1/\lambda}\right]^{\lambda/\lambda_2}. \tag{61}$$

The constant of integration Λ could be determined by imposing a boundary condition. Without any loss of generality, we assume that at $x=1$, $w_0=0$ giving

$$\Lambda=\sigma(\sigma\delta)^{1/\lambda}\frac{\lambda_2}{\lambda_1}\Delta^{-\delta\lambda_1/\lambda}, \tag{62}$$

which leads to

$$w=\tau^{-\delta(n+1)}\left[\sigma^{\lambda+1}\delta\left(\frac{\lambda_2}{\lambda_1}\right)^{\lambda}\right]^{\lambda/\lambda_2}\left[\Delta^{-\delta\lambda_1/\lambda}-\left(\frac{x}{\tau^{\delta}}\right)^{\lambda_1/\lambda}\right]^{\lambda/\lambda_2}. \tag{63}$$

It can be easily verified that Eq. (63), with $\sigma=1$, is equivalent to the solution (31) and (32) (obtained in Sec. III A) corresponding to the spread of a constant volume of liquid.

It is of interest that the solution displayed in Eq. (60) can be transformed into the waiting-time solution already discussed in Sec. III B. For $\tau\to 0$ (or $t\to-\Delta$), Λ can be easily neglected in Eq. (60) yielding

$$w(\tau\to 0)=\left[-\left(\frac{\lambda_2}{\lambda_1}\right)\frac{\delta x^{\lambda_1}}{t}\right]^{1/\lambda_2}, \tag{64}$$

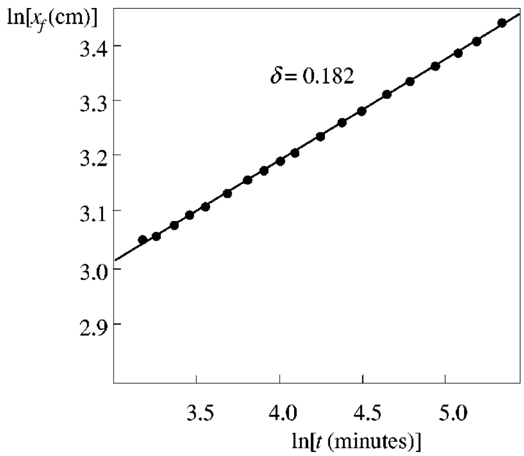

FIG. 2. Self-similar motion of the front of a waiting-time current for $t > t_w$ (from Ref. [4]).

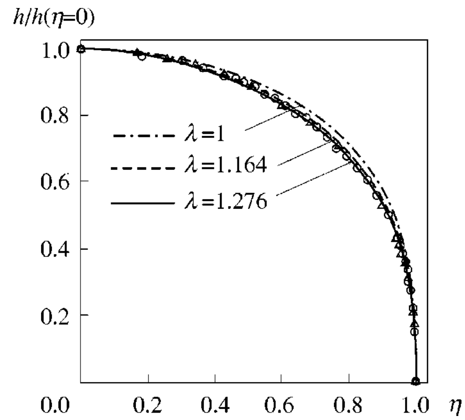

FIG. 3. Theoretical (this work) and experimental [14] profiles of self-similar constant-volume currents. Triangles and circles represent measurements at two different times, both in the self-similar regime. The theoretical profiles correspond to $\lambda = 1$ (Newtonian liquid), $\lambda = 1.164$ (which is the rheological index that gives the best fit of the motion of the front, see Fig. 2), and $\lambda = 1.276$ [which gives the best fit of the profile of the current, Eq. (31)].

which is precisely the solution given by Eq. (35). Notice that in this derivation the solution is valid for either positive ($\Delta = -|\Delta|$) or negative τ, but it blows up at $\tau = 0$ ($t = \pm|\Delta|$). Of course, the waiting-time solution (35) can be seen as the special solution with the constant of integration $\Lambda = 0$.

IX. DISCUSSION AND FINAL REMARKS

Based on a generalization of the lubrication approximation [11], we have derived the governing equations that describe creeping gravity currents of non-Newtonian liquids having a power-law rheology. Currents that depend on a single horizontal coordinate have been considered, both in Cartesian and in axial symmetry. The equations are of a nonlinear parabolic type, and differ from those for Newtonian liquids in that they are nonlinear in the spatial derivative of the thickness of the current. However, many properties of the solutions are closely analogous to those for Newtonian rheology [7,8]. In particular, the spreading relations for the currents can also be expressed as power laws in time. The exponents, however, are functions of the rheological index, and thus differ from those corresponding to Newtonian liquids. The similarity solutions corresponding to currents whose volume varies as a power of time have been investigated. For the spread of a constant volume of liquid, analytic solutions are obtained both for the Cartesian and for the axial symmetry. Solutions of the waiting-time type are found, as well as steady flows from a constant source to a sink at a fixed position. General traveling-wave solutions have also been obtained in closed form, and analytic formulas for a simple case are given. A general phase plane formalism, which allows us to systematically investigate the entire family of self similar solutions, is introduced. The application of the Boltzmann transform for solving the governing equations is briefly discussed.

There is a good agreement between the present theory and some experiments [4,14]. This can be appreciated in Figs. 2 and 3, which are based on measurements [4,14] of constant-volume waiting-time currents with plane symmetry ($n = 0$). A silicone putty (Rodhorsil Gomme spéciale GSIR, manufactured by Rhône-Poulenc, France) loaded with sand was used. The rheological behavior of this putty was investigated in [29] by means of a rotational viscometer; these measurements show a Newtonian behavior for large strain rates ($10^{-2}\ \text{s}^{-1}$ or larger), while for very small strain rates ($10^{-5}\ \text{s}^{-1}$ or less) the behavior can be approximated by Eq. (1) with $\lambda \simeq 1.6$. The experiments of [4,14] were performed during our research on waiting-time currents (see [5]) but the use of the silicone putty was discontinued when we realized that its behavior was not Newtonian, and the results were not included in the cited reference. These measurements correspond to strain rates in the range between 10^{-4} and $10^{-3}\ \text{s}^{-1}$, and the results of [29] suggest that in this intermediate range a power-law rheology as Eq. (1) can be used, with λ around 1.2–1.3 [see Fig. (9) of [29]].

We shall briefly describe the experiments of [4,14]; the fluid was contained in a rectangular perspex tray and the initial condition consisted of a wedge-shaped profile ($h \propto X$), which according to the discussion of Sec. VII yields a waiting-time flow. The position of the front as well as the thickness profile of the current were measured. To this purpose the authors employed a sheet of laser light obtained by means of a slit and a cylindrical lens, to produce a well defined, narrow line on the surface of the current, whose image was recorded and analyzed. This experimental setup allowed us to measure the profile within an error of 0.3 mm. As expected, after the fluid was released, a time interval t_w ensued during which the profile of the current changed but the front remained motionless. At $t = t_w$, the front started to move. In Fig. 2 it can be observed that for $t > t_w$, the motion of the front follows very closely Eq. (27), with $\delta = 0.182$, which corresponds to a rheological index $\lambda = 1.164$. In Fig. 3 we show the profiles of the current for two different moments, within the self-similar regime, as well as the theoretical profiles [Eq. (31)] corresponding to $\lambda = 1$, $\lambda = 1.164$, and $\lambda = 1.276$ (the latter is the value that gives the best fit of the experimental points and would result in $\delta = 0.171$); however, the difference between these values is not significant in view of the experimental uncertainties. In conclusion, we believe that there is a reasonable consistency between the theoretical profile corresponding to the rheological index derived from the dynamics (Fig. 2) of the current and the measured profile, and both are consistent with the independent determination of λ of Ref. [29], which lends support to the theory presented here. It can also be observed that in the experiments considered here, the difference between the Newtonian and the non-Newtonian profiles is small (λ is close to unity), but significant.

The analogy between the present results and those derived for Newtonian liquids (all of our solutions have their counterparts in Newtonian rheology) can be traced to the fact that the constitutive relation (1) introduces a single-dimensional parameter (A) into the problem, as happens in the case of a Newtonian liquid (the viscosity coefficient). In both instances, this dimensional parameter can be scaled out by an appropriate definition of the dependent variable, and thus does not appear in the final governing equations. For this reason one finds similarity solutions whenever the analogous Newtonian problem is self-similar. The dimensionality of A depends on the rheological index λ and, as a consequence, the spreading relations have rheology-dependent exponents. It is interesting to observe that in most cases this dependence is rather weak, a fact that was already pointed out in a particular instance [16]. However, the differences between Newtonian and non-Newtonian currents are significant and can be clearly observed in the experiments, notwithstanding that these experiments were not specifically designed to test the theory.

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